DATE : 03/10/2021

Time : 3 hrs. **Max. Marks: 180**

Answers & Solutions

for

JEE (Advanced)-2021

PAPER - 2

PART-I : PHYSICS

SECTION - 1

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If only (all) the correct option(s) is(are) chosen;

  Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;

  Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;

  Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

  Zero Marks : 0 If unanswered;

  Negative Marks : -2 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

  choosing ONLY (A), (B) and (D) will get +4 marks;

  choosing ONLY (A) and (B) will get +2 marks;

  choosing ONLY (A) and (D) will get +2 marks;

  choosing ONLY (B) and (D) will get +2 marks;

  choosing ONLY (A) will get +1 mark;

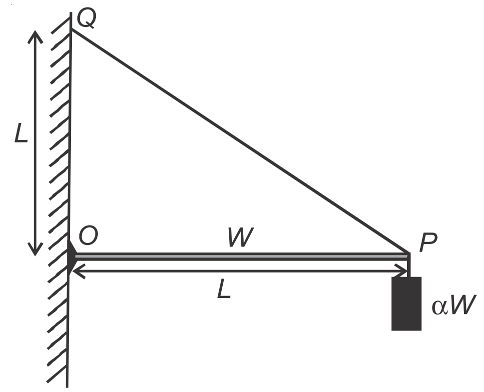
  choosing ONLY (B) will get +1 mark;

  choosing ONLY (D) will get +1 mark;

  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and

  choosing any other option(s) will get -2 marks.

1. One end of a horizontal uniform beam of weight W and length L is hinged on a vertical wall at point O and its other end is supported by a light inextensible rope. The other end of the rope is fixed at point Q, at a height L above the hinge at point O. A block of weight αW is attached at the point P of the beam, as shown in the figure (not to scale). The rope can sustain a maximum tension of $(2\sqrt{2})W$. Which of the following statement(s) is(are) correct?



(A) The vertical compenent of reaction force at O does **not** depend on α

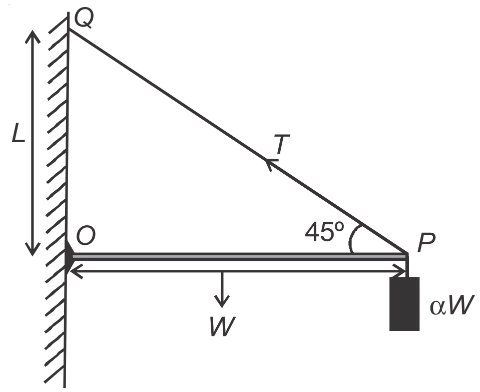
(B) The horizontal component of reaction force at O is equal to W for $\alpha = 0.5$

(C) The tension in the rope is $2W$ for $\alpha = 0.5$

(D) The rope breaks if $\alpha > 1.5$

Answer (A, B, D)

Sol. $W \times \frac{L}{2} + \alpha W \times L = T \times \frac{1}{\sqrt{2}} \times L$



$\Rightarrow \quad T = \sqrt{2}\left(\frac{1}{2} + \alpha\right)W$

$\therefore \quad T \times \frac{1}{\sqrt{2}} + F_V = W + \alpha W$

$\Rightarrow \quad \frac{W}{2} + \alpha W + F_V = W + \alpha W$

$\Rightarrow \quad F_V = \frac{W}{2}$

At $\alpha = \frac{1}{2}$, $\quad T = \sqrt{2}\left(\frac{1}{2} + \frac{1}{2}\right)W = \sqrt{2}W$

$F_H\left(at \;\; \alpha = \frac{1}{2}\right) = \sqrt{2}W \times \frac{1}{\sqrt{2}} = W$

at $\alpha = 1.5$, $T = \sqrt{2} \times \left(\frac{1}{2} + \frac{3}{2}\right)W = 2\sqrt{2}W$

2. A source, approaching with speed u towards the open end of a stationary pipe of length L, is emitting a sound of frequency f_s. The farther end of the pipe is closed. The speed of sound in air is v and f_0 is the fundamental frequency of the pipe. For which of the following combination(s) of u and f_s, will the sound reching the pipe lead to a resonance?

(A) $u = 0.8v$ and $f_s = f_0$

(B) $u = 0.8v$ and $f_s = 2f_0$

(C) $u = 0.8v$ and $f_s = 0.5f_0$

(D) $u = 0.5v$ and $f_s = 1.5f_0$

Answer (A, D)

Sol. For resonance,

$$\frac{v}{v-u}\times f_s = (\text{odd})\times f_0$$

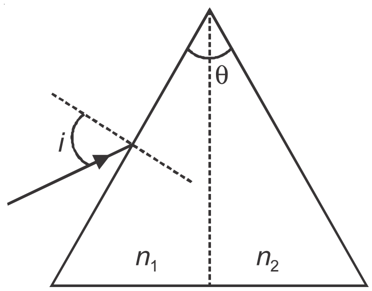
(A) $\frac{v}{v-0.8v}\times f_0 = 5f_0$

(B) $\frac{v}{v-0.8v}\times 2f_0 = 10f_0$

(C) $\frac{v}{v-0.8v}\times\left(\frac{f_0}{2}\right) = \frac{5}{2}f_0$

(D) $\frac{v}{v-0.5v}\times\left(\frac{3f_0}{2}\right) = 3f_0$

3. For a prism of prism angle $\theta = 60^\circ$, the refractive indices of the left half and the right half are, respectively, n_1 and n_2 ($n_2 \ge n_1$) as shown in the figure. The angle of incidence i is chosen such that the incident light rays will have minimum deviation if $n_1 = n_2 = n = 1.5$. For the case of unequal refractive indices, $n_1 = n$ and $n_2 = n + \Delta n$ (where $\Delta n << n$), the angle of emergence $e = i + \Delta e$. Which of the following statement(s) is(are) correct?



(A) The value of Δe (in radians) is greater than that of Δn

(B) Δe is proportional to Δn

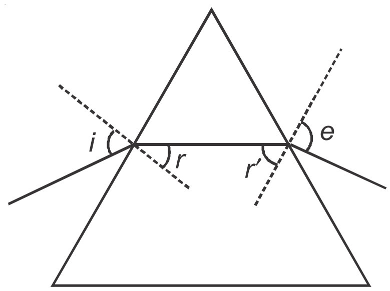
(C) Δe lies between 2.0 and 3.0 milliradians, if $\Delta n = 2.8 \times 10^{-3}$

(D) Δe lies between 1.0 and 1.6 milliradians, if $\Delta n = 2.8 \times 10^{-3}$

Answer (B, C)

Sol. For $n_1 = n_2 = n = 1.5$,

$r = 30^\circ$



$\therefore$ $\sin i = 1.5 \times \sin(30^\circ) = \frac{3}{4}$

$\Rightarrow$ $\sin e = \frac{3}{4}$ for $n_1 = n_2$

Now, $r' = 30^\circ$ and $n_2 = n + \Delta n$

$$(n_2)\times\sin(r') = 1\times\sin e$$

$$\Rightarrow \Delta n_2\times\sin(30^\circ) = \cos e\times\Delta e$$

$$\Rightarrow \quad \Delta e = \frac{(\Delta n)\times\frac{1}{2}}{\sqrt{1-\frac{9}{16}}} = \frac{2}{\sqrt{7}}\Delta n$$

$\Rightarrow \quad \Delta e < \Delta n$ and, $\Delta e \propto \Delta n$

at $\Delta n = 2.8 \times 10^{-3}$, $\Delta e = 2.12 \times 10^{-3}$ rad

4. A physical quantity $\vec{S}$ is defined as $\vec{S} = (\vec{E}\times\vec{B})/\mu_0$, where $\vec{E}$ is electric field, $\vec{B}$ is magnetic field and μ_0 is the permeability of free space. The dimensions of $\vec{S}$ are the same as the dimensions of which of the following quantity(ies)?

(A) $\dfrac{\text{Energy}}{\text{Charge}\times\text{Current}}$

(B) $\dfrac{\text{Force}}{\text{Length}\times\text{Time}}$

(C) $\dfrac{\text{Energy}}{\text{Volume}}$

(D) $\dfrac{\text{Power}}{\text{Area}}$

Answer (B, D)

Sol. $\vec{S} = \dfrac{\vec{E}\times\vec{B}}{\mu_0}$

$\vec{S}$ is known as poynting vector and represents intensity of electromagnetic waves.

$$\left[\vec{S}\right] = [MT^{-3}] = \left[\frac{\text{Power}}{\text{Area}}\right] = \left[\frac{\text{Force}}{\text{Length}\times\text{Time}}\right]$$

5. A heavy nucleus N, at rest, undergoes fission $N \to P + Q$, where P and Q are two lighter nuclei. Let $\delta = M_N - M_P - M_Q$, where M_P, M_Q and M_N are the masses of P, Q and N, respectively. E_P and E_Q are the kinetic energies of P and Q, respectively. The speeds of P and Q are V_P and V_Q, respectively. If c is the speed of light, which of the following statement(s) is(are) correct?

(A) $E_P + E_Q = c^2\delta$

(B) $E_P = \left(\dfrac{M_P}{M_P+M_Q}\right)c^2\delta$

(C) $\dfrac{V_P}{V_Q} = \dfrac{M_Q}{M_P}$

(D) The magnitude of momentum for P as well as Q is $c\sqrt{2\mu\delta}$, where $\mu = \dfrac{M_P M_Q}{(M_P+M_Q)}$

Answer (A, C, D)

Sol. $E_P + E_Q = \delta c^2$ (Q-value of nuclear reaction)

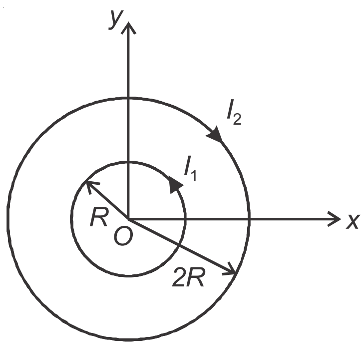
$\sqrt{2M_P E_P} = \sqrt{2M_Q E_Q}$ or $M_P V_P = M_Q V_Q$

$\Rightarrow \dfrac{E_P}{E_Q} = \dfrac{M_Q}{M_P}$

$\Rightarrow E_P = \dfrac{M_Q}{M_P + M_Q}\delta c^2$

$\Rightarrow$ Momentum of P or Q = $\sqrt{\dfrac{2M_P M_Q}{M_P + M_Q}\delta c^2}$

6. Two concentric circular loops, one of radius R and the other of radius $2R$, lie in the xy-plane with the origin as their common center, as shwon in the figure. The smaller loop carries current I_1 in the anti-clockwise direction and the larger loop carries current I_2 in the clockwise direction, with $I_2 > 2I_1$. $\vec{B}(x, y)$ denotes the magnetic field at a point (x, y) in the xy-plane. Which of the following statement(s) is(are) correct?



(A) $\vec{B}(x, y)$ is perpendicular to the xy-plane at any point in the plane

(B) $|\vec{B}(x, y)|$ depends on x and y only through the radial distance $r = \sqrt{x^2 + y^2}$

(C) $|\vec{B}(x, y)|$ is non-zero at all points for $r < R$

(D) $\vec{B}(x, y)$ points normally outward from the xy-plane for all the points between the two loops

Answer (A, B)

Sol. Magnetic field due to a circular loop at any point in its plane will be perpendicular to the plane. Due to symmetry it will depend only on distance from centre. Field will be in opposite direction inside and outside the loop.

Field may be non-zero for $r < R$, as it is in opposite direction due to both the loops.

SECTION - 2

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

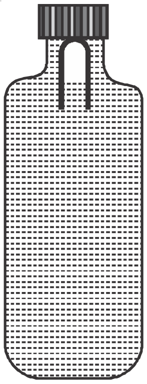
Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 7 and 8

Question Stem

A soft plastic bottle, filled with water of density 1 gm/cc, carries an inverted glass test-tube with some air (ideal gas) trapped as shown in the figure. The test-tube has a mass of 5 gm, and it is made of a thick glass of density 2.5 gm/cc. Initially the bottle is sealed at atmospheric pressure $p_0 = 10^5$ Pa so that the volume of the trapped air is $V_0 = 3.3$ cc. When the bottle is squeezed from outside at constant temperature, the pressure inside rises and the volume of the trapped air reduces. It is found that the test tube begins to sink at pressure $p_0 + \Delta p$ without changing its orientation. At this pressure, the volume of the trapped air is $V_0 - \Delta V$.

Let $\Delta V = X$ cc and $\Delta p = Y \times 10^3$ Pa.



7. The value of X is ___________.

Answer (0.30)

8. The value of Y is ___________.

Answer (10.00)

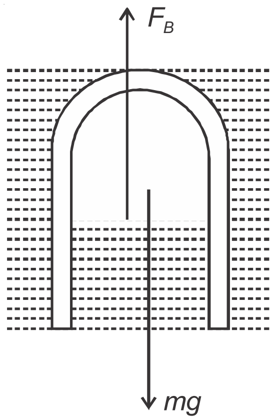
Solution of Q. Nos. 7 & 8

When bouyant force on (tube + air) system will become equal to weight of tube then tube will start sinking. (Here we can neglect weight of air as compared to weight of tube)

Now, Let volume of air in this case = V_{air}

$F_B = mg$

So, $\delta_w (V_{tube} + V_{air})\, g = mg$



$\Rightarrow \quad 1\left(\frac{5}{2.5}\text{ cm}^3 + V_{air}\right) = 5$

$\Rightarrow \quad 2 + V_{air} = 5$

$V_{air} = 3\text{ cm}^3$

As initial volume of air = 3.3 cm^3

So, $\Delta V = 0.3$ cc

So, $X = 0.30$

As temperature of air is constant

So, PV = constant

$P_0\, 3.3 = P_f\, 3$, P_f is final pressure of air

$\Rightarrow \quad P_f = 1.1\, P_0 = P_0 + 0.1\, P_0$

So, $\Delta P = 10^4$ Pa

So, $Y = 10$

So, $X = 0.30$

$Y = 10.00$

Question Stem for Question Nos. 9 and 10

Question Stem

A pendulum consists of a bob of mass m = 0.1 kg and a massless inextensible string of length L = 1.0 m. It is suspended from a fixed point at height H = 0.9 m above a frictionless horizontal floor. Initially, the bob of the pendulum is lying on the floor at rest vertically below the point of suspension. A horizontal impulse P = 0.2 kg-m/s is imparted to the bob at some instant. After the bob slides for some distance, the string becomes taut and the bob lifts off the floor. The magnitude of the angular momentum of the pendulum about the point of suspension just before the bob lifts off is J kg-m^2/s. The kinetic energy of the pendulum just after the lift-off is K Joules.

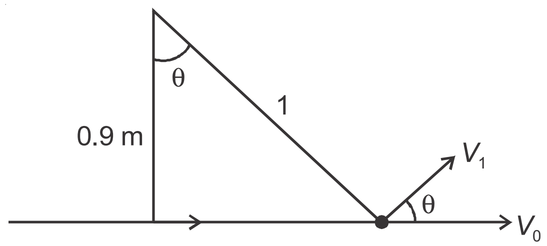
9. The value of J is __________.

Answer (0.18)

10. The value of K is __________.

Answer (0.16)

Solution of Q. Nos. 9 & 10



$$V_0 = \frac{0.2}{0.1} = 2 \text{ m/s}$$

9. $L = P \times H$

$= 0.2 \times 0.9$

$= 0.18$ kg m^2/s

10. $V_1 = V_0 \cos\theta = 2 \times \left(\frac{0.9}{1}\right)$

$\therefore \quad K = \frac{1}{2} \times (0.1) \times (2 \times 0.9)^2 = 0.162$ Joules

Question Stem for Question Nos. 11 and 12

Question Stem

In a circuit, a metal filament lamp is connected in series with a capacitor of capacitance C μF across a 200 V, 50 Hz supply. The power consumed by the lamp is 500 W while the voltage drop across it is 100 V. Assume that there is no inductive load in the circuit. Take *rms* values of the voltages. The magnitude of the phase-angle (in degrees) between the current and the supply voltage is φ. Assume, $\pi\sqrt{3} \approx 5$.

11. The value of C is ___.

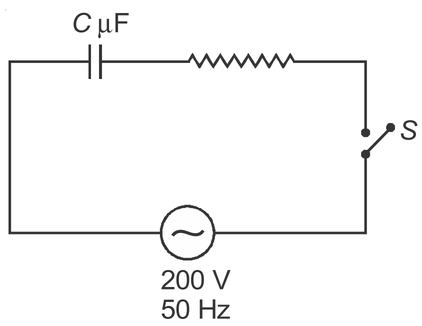
Answer (100)

Sol.

12. The value of φ is ___.

Answer (60)

Solution of Q. Nos. 11 & 12



$$P = \frac{V^2}{R} \Rightarrow 500 = \frac{100^2}{R}$$

$\Rightarrow$ R = 20 Ω

Now across resistance

500 = I × 100

$\Rightarrow$ I_{rms} = 5 A

V_{rms} = 200 V, $V_{rms/real}$ = 100 V

$$\cos\phi = \frac{100}{200} = \frac{1}{2} \Rightarrow \phi = 60^\circ$$

$$\tan\phi = \frac{X_C}{R} = \frac{1}{\omega RC}$$

$$\sqrt{3} = \frac{1}{100\pi(20)C}$$

$$\Rightarrow C = \frac{1}{20\pi\sqrt{3}\times 100} = 10^{-4}\text{ F} = 100\ \mu\text{F}$$

SECTION - 3

- This section contains **TWO (02) paragraphs.** Based on each paragraph, there are **TWO (02)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : Full Marks : +3 If ONLY the correct option is chosen;

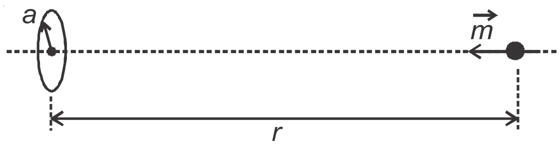
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

  Negative Marks : -1 In all other cases.

Paragraph

A special metal S conducts electricity without any resistance. A closed wire loop, made of S, does not allow any change in flux through itself by inducing a suitable current to generate a compensating flux. The induced current in the loop cannot decay due to its zero resistance. This current gives rise to a magnetic moment which in turn repels the source of magnetic field or flux. Consider such a loop, of radius a, with its center at the origin. A magnetic dipole of moment m is brought along the axis of this loop from infinity to a point at distance r (>> a) from the center of the loop with its north pole always facing the loop, as shown in the figure below.

The magnitude of magnetic field of a dipole m, at a point on its axis at distance r, is $\frac{\mu_0 m}{2\pi r^3}$, where μ_0 is the permeability of free space. The magnitude of the force between two magnetic dipoles with moments, m_1 and m_2, separated by a distance r on the common axis, with their north poles facing each other, is $\frac{km_1m_2}{r^4}$, where k is a constant of appropriate dimensions. The direction of this force is along the line joining the two dipoles.



13. When the dipole m is placed at a distance r from the center of the loop (as shown in the figure), the current induced in the loop will be proportional to

(A) m/r^3

(B) m^2/r^2

(C) m/r^2

(D) m^2/r

Answer (A)

Sol. Magnetic flux due to dipole through ring = $\frac{\mu_0}{2\pi}\times\frac{m}{r^3}\times\pi a^2$

for net magnetic flux through the loop to be zero.

Magnetic flux due to dipole = Magnetic flux due to induced current

$\Rightarrow \frac{\mu_0}{2\pi}\times\pi a^2\times\frac{m}{r^3} = I\times\pi a^2\times\frac{k}{a}$, where k is proportionality constant.

$\Rightarrow I \propto \frac{m}{r^3}$

14. The work done in bringing the dipole from infinity to a distance r from the center of the loop by the given process is proportional to

(A) m/r^5

(B) m^2/r^5

(C) m^2/r^6

(D) m^2/r^7

Answer (C)

Sol. $F = \frac{km_1m_2}{r^4} = k\left(I\pi a^2\right)\left(\frac{m}{r^4}\right)$

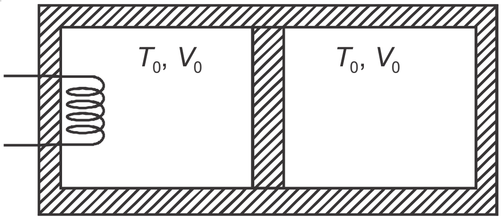
$F = C\frac{m^2}{r^7}$ where C is a constant obtained by substituting the value of I from Q.13

$|W| = \int_{\infty}^{r} F\,dr = Cm^2\int_{\infty}^{r}\frac{dr}{r^7} = \frac{C'm^2}{r^6}$ where C' is a constant

$|W| \propto \frac{m^2}{r^6}$

Paragraph

A thermally insulating cylinder has a thermally insulating and frictionless movable partition in the middle, as shown in the figure below. On each side of the partition, there is one mole of an ideal gas, with specific heat at constant volume, $C_V = 2R$. Here, R is the gas constant. Initially, each side has a volume V_0 and temperature T_0. The left side has an electric heater, which is turned on at very low power to transfer heat Q to the gas on the left side. As a result the partition moves slowly towards the right reducing the right side volume to $V_0/2$. Consequently, the gas temperatures on the left and the right sides become T_L and T_R, respectively. Ignore the changes in the temperatures of the cylinder, heater and the partition.



15. The value of $\frac{T_R}{T_0}$ is

(A) $\sqrt{2}$

(B) $\sqrt{3}$

(C) 2

(D) 3

Answer (A)

Sol. $PV^{\gamma} = C$

$\Rightarrow TV^{\gamma-1} = C$

$$\Rightarrow T_0V_0^{\gamma-1} = T_R\left(\frac{V_0}{2}\right)^{\gamma-1}$$

$$C_V = \frac{R}{\gamma-1}$$

$$\Rightarrow 2R = \frac{R}{\gamma-1}$$

$$\Rightarrow \gamma-1 = \frac{1}{2}$$

$$\Rightarrow \gamma = \frac{3}{2}$$

$$\Rightarrow \frac{T_R}{T_0} = 2^{\gamma-1} = \sqrt{2}$$

16. The value of $\frac{Q}{RT_0}$ is

(A) $4(2\sqrt{2}+1)$

(B) $4(2\sqrt{2}-1)$

(C) $(5\sqrt{2}+1)$

(D) $(5\sqrt{2}-1)$

Answer (B)

Sol. $Q = \Delta U_1 + \Delta U_2$

$\Delta U_1 = C_V \Delta T_1 = 2R(T_L - T_0)$

$\Delta U_2 = C_V \Delta T_2 = 2R(T_R - T_0)$

$T_L = 3\sqrt{2}T_0$, $T_R = \sqrt{2}T_0$

$$Q = 2R\left[3\sqrt{2} - 1\right]T_0 + 2R(\sqrt{2} - 1)T_0$$

$$Q = 4RT_0[2\sqrt{2} - 1]$$

$$\Rightarrow \frac{Q}{RT_0} = 4[2\sqrt{2} - 1]$$

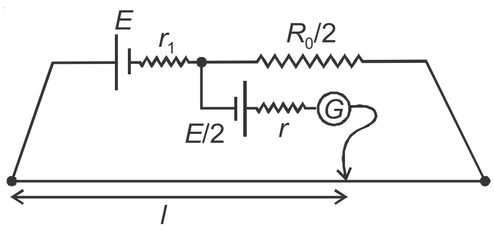
SECTION - 4

- This section contains THREE (03) questions.
- The answer to each question is a NON-NEGATIVE INTEGER.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct integer is entered;

Zero Marks : 0 In all other cases.

17. In order to measure the internal resistance r_1 of a cell of emf E, a meter bridge of wire resistance $R_0 = 50\ \Omega$, a resistance $R_0/2$, another cell of emf $E/2$ (internal resistance r) and a galvanometer G are used in a circuit, as shown in the figure. If the null point is found at $l = 72$ cm, then the value of r_1 = ___ Ω.



Answer (3)

Sol. Current will flow in main circuit

$$I = \frac{E}{r_1 + \frac{3R_0}{2}}$$

$$+E - IR_0 \times 0.72 - Ir_1 - \frac{E}{2} = 0$$

$$\frac{E}{2} = \frac{2E}{2r_1 + 3R_0} \times \left[0.72R_0 + r_1\right]$$

$$2r_1 + 3R_0 = 4\left[0.72R_0 + r_1\right]$$

$0.12R_0 = 2r_1$

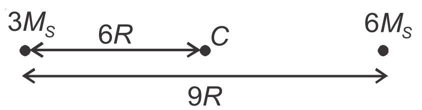
$r_1 = 3\Omega$

18. The distance between two stars of masses $3M_s$ and $6M_s$ is $9R$. Here R is the mean distance between the centers of the Earth and the Sun, and M_s is the mass of the Sun. The two stars orbit around their common center of mass in circular orbits with period nT, where T is the period of Earth's revolution around the Sun.

The value of n is ____.

Answer (9)

Sol.



Centre of mass of system lies at 6R from lighter mass.

$$\left[3M_S\omega^2 \times 6R\right] = \frac{G(18M_S^2)}{81R^2}$$

$$\omega^2 R = \frac{GM_S}{81R^2}$$

$$T' = \sqrt{\frac{81R^3}{GM_S}}$$

$$T' = 9T$$

n = 09

19. In a photoemission experiment, the maximum kinetic energies of photoelectrons from metals P, Q and R are E_P, E_Q and E_R, respectively, and they are related by $E_P = 2E_Q = 2E_R$. In this experiment, the same source of monochromatic light is used for metals P and Q while a different source of monochromatic light is used for the metal R. The work functions for metals P, Q and R are 4.0 eV, 4.5 eV and 5.5 eV, respectively. The energy of the incident photon used for metal R, in eV, is ____.

Answer (6)

Sol. $$\frac{hc}{\lambda_1} = \varphi_P + E_P$$

$$\frac{hc}{\lambda_1} = \varphi_Q + E_Q$$

$E_P = 2E_Q$

$E_P - E_Q = 0.5$

$\Rightarrow$ $E_P = 1.0$ eV, $E_Q = 0.5$ eV

$E_R = 0.5$ eV

Energy of incident photon on $R = \varphi_R + E_R = 6$ eV

PART-II : CHEMISTRY

SECTION - 1

- This section contains **EIGHT (08)** questions.
- Each question has **FOUR** options (A), (B), (C) & (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If only (all) the correct option(s) is(are) chosen;
Partial Marks	:	+3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	:	+2	If three or more options are correct but ONLY two options are chosen, and both of which are correct;
Partial Marks	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	–2	In all other cases.

- For example : in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

  choosing ONLY (A), (B) and (D) will get +4 marks;

  choosing ONLY (A) and (B) will get +2 marks;

  choosing ONLY (A) and (D) will get +2 marks;

  choosing ONLY (B) and (D) will get +2 marks;

  choosing ONLY (A) will get +1 mark;

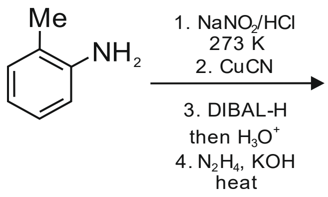
  choosing ONLY (B) will get +1 mark;

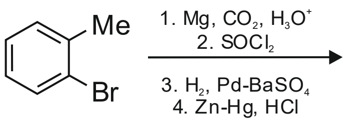
  choosing ONLY (D) will get +1 mark;

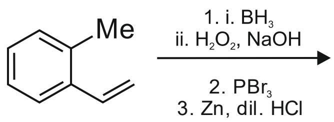
  choosing no option (i.e., the question is unanswered) will get 0 marks; and

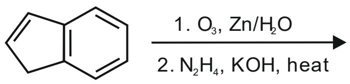
  choosing any other combination of options will get –2 mark.

1. The reaction sequence(s) that would lead to *o*-xylene as the major product is(are)

(A) 

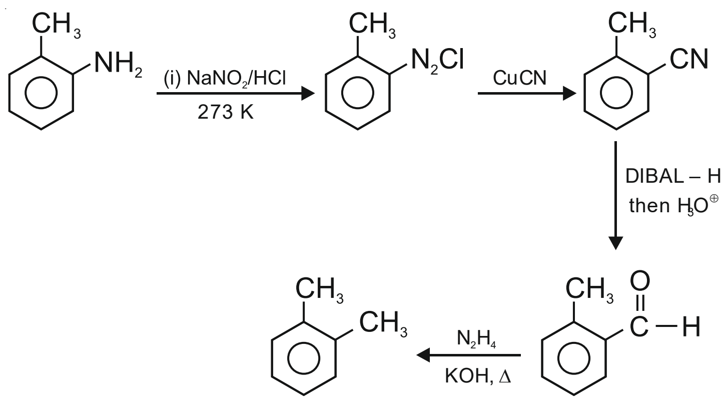
(B) 

(C) 

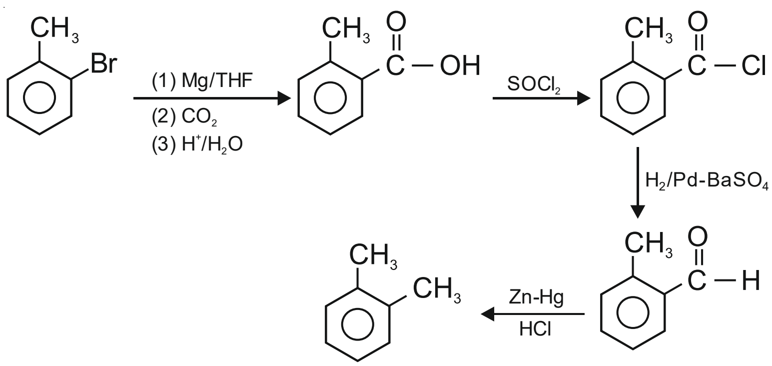
(D) 

Answer (A, B)

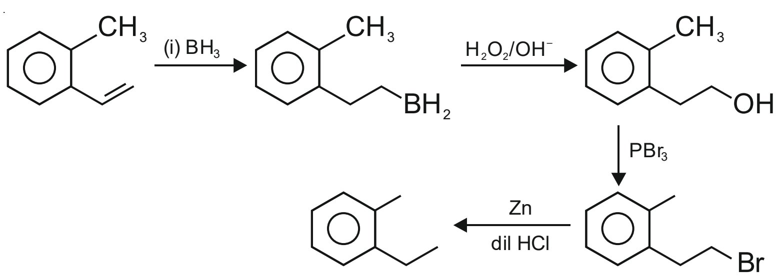
Sol. (A)



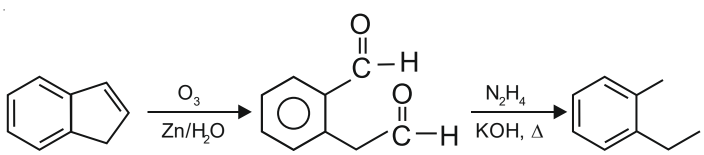
(B)



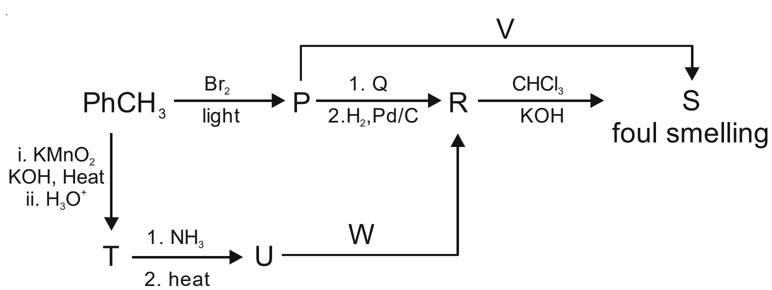
(C)



(D)



2. Correct option(s) for the following sequence of reactions is(are)



(A) $Q = KNO_2$, $W = LiAlH_4$

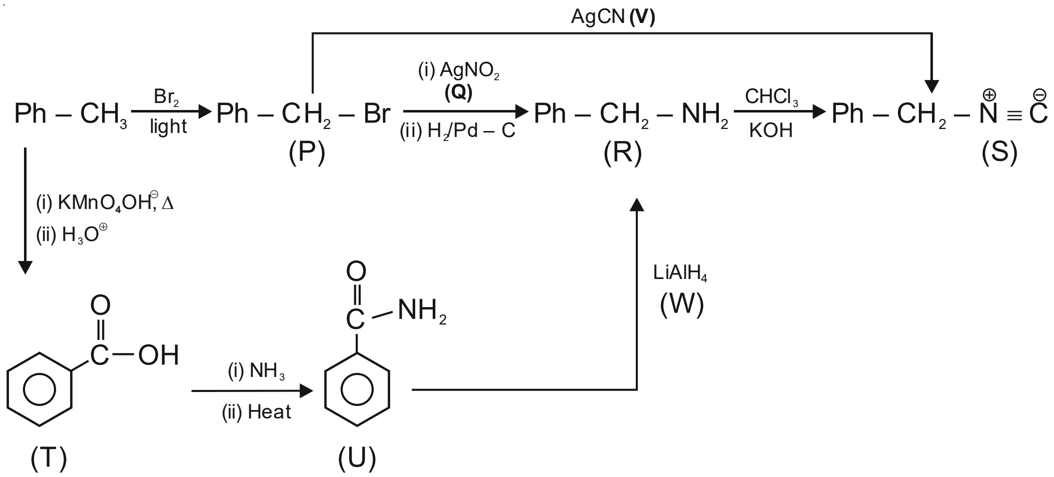
(B) R = benzenamine, V = KCN

(C) $Q = AgNO_2$, R = phenylmethanamine

(D) $W = LiAlH_4$, $V = AgCN$

Answer (C, D)

Sol.



∴ Correct option are

$Q = AgNO_2$, R = phenylmethanamine

$W = LiAlH_4$, V = AgCN

3. For the following reaction

$$2X + Y \xrightarrow{k} P$$

the rate of reaction is $\frac{d[P]}{dt} = k[X]$. Two moles of X are mixed with one mole of Y to make 1.0 L of solution. At 50 s, 0.5 mole of Y is left in the reaction mixture. The correct statement(s) about the reaction is(are)

(Use: ln 2 = 0.693)

(A) The rate constant, k, of the reaction is $13.86 \times 10^{-4}\ s^{-1}$.

(B) Half-life of X is 50 s.

(C) At 50 s, $-\frac{d[X]}{dt} = 13.86 \times 10^{-3}\ mol\ L^{-1}\ s^{-1}$.

(D) At 100 s, $-\frac{d[Y]}{dt} = 3.46 \times 10^{-3}\ mol\ L^{-1}\ s^{-1}$.

Answer (B,C,D)

Sol. $\text{rate} = \frac{d[P]}{dt} = k[X]$

2X	+	Y	→	P
2 mole		1 mole		
1 mole		0.5 mole		0.5 mole

$$-\frac{d[X]}{dt} = k_1[X] = 2k[X] \Rightarrow 2k = k_1$$

$$-\frac{d[Y]}{dt} = k_2[X] = k[X] \Rightarrow k_2 = k$$

$$2k = \frac{1}{50}\ln 2$$

$$k = \frac{1}{100}\ln 2 = \frac{0.693}{100} = 6.93 \times 10^{-3}\ s^{-1}$$

$$(t_{1/2})_x = \frac{\ln 2}{k_1} = \frac{\ln 2 \times 100}{2 \times 0.693} = 50\ \text{sec}$$

At 50 sec

$$-\frac{d[X]}{dt} = 2k[X] = 2 \times \frac{0.693}{100} \times 1$$

$= 13.86 \times 10^{-3}$ mol L^{-1} s^{-1}

At 100 sec

$$-\frac{d[Y]}{dt} = k_2[X] = k[X] = \frac{0.693}{100} \times \frac{1}{2}$$

($\because$ Concentration of X after 2 half lives = $\frac{1}{2}$ M)

$= 3.46 \times 10^{-3}$ mol L^{-1} s^{-1}

4. Some standard electrode potentials at 298 K are given below:

Pb^{2+}/Pb	–0.13 V
Ni^{2+}/Ni	–0.24 V
Cd^{2+}/Cd	–0.40 V
Fe^{2+}/Fe	–0.44 V

To a solution containing 0.001 M of X^{2+} and 0.1 M of Y^{2+}, the metal rods X and Y are inserted (at 298 K) and connected by a conducting wire. This resulted in dissolution of X. The correct combination(s) of X and Y, respectively, is(are)

(Given: Gas constant, R = 8.314 J K^{-1} mol^{-1}, Faraday constant, F = 96500 C mol^{-1})

(A) Cd and Ni
(B) Cd and Fe
(C) Ni and Pb
(D) Ni and Fe

Answer (A,B,C)

Sol. $X + Y^{2+} \rightarrow X^{2+} + Y$

$$E = E^\circ - \frac{0.06}{2}\log_{10}\left(\frac{10^{-3}}{10^{-1}}\right)$$

$E = E^\circ + 0.06$

(A) $E^\circ = -(-.4) + (-.24) = .16 > 0$

(B) $E^\circ = -(-.4) + (-.44) = -.04 < 0$ and $E_{cell} = -0.04 + 0.06 = +0.02 > 0$

(C) $E^\circ = -(-.24) + (-.13) = .11 > 0$

(D) $E^\circ = -(-.24) + (-.44) = -.2 < 0$

$\therefore$ $E_{cell} = -0.2 + 0.06 = -0.14 < 0$

$\therefore$ If $E_{cell} > 0$ then the cell construction is possible.

5. The pair(s) of complexes wherein both exhibit tetrahedral geometry is(are)

(Note: py = pyridine

Given: Atomic numbers of Fe, Co, Ni and Cu are 26, 27, 28 and 29, respectively)

(A) $[FeCl_4]^-$ and $[Fe(CO)_4]^{2-}$

(B) $[Co(CO)_4]^-$ and $[CoCl_4]^{2-}$

(C) $[Ni(CO)_4]$ and $[Ni(CN)_4]^{2-}$

(D) $[Cu(py)_4]^+$ and $[Cu(CN)_4]^{3-}$

Answer (A,B,D)

Sol. $[FeCl_4]^- \to Fe^{3+}$, $3d^5$ (weak field ligand) = sp^3

$[Fe(CO)_4]^{-2} \to Fe^{2-}$, $3d^{10} \to sp^3$

$[Co(CO)_4]^- \to Co^-$, $3d^{10} \to sp^3$

$[CoCl_4]^{2-} \to Co^{2+}$, $3d^7$ (weak field ligand) $\to sp^3$

$[Ni(CO)_4] \to Ni$, $3d^{10} \to sp^3$

$[Ni(CN)_4]^{2-} \to Ni^{2+}$, $3d^8$ (strong field ligand) $\to dsp^2$

$[Cu(py)_4]^+ \to Cu^+$, $3d^{10} \to sp^3$

$[Cu(CN)_4]^{3-} \to Cu^+$, $3d^{10} \to sp^3$

In $3d^{10}$ electronic configuration only sp^3 hybridisation and tetrahedral geometry is possible.

6. The correct statement(s) related to oxoacids of phosphorous is(are)

(A) Upon heating, H_3PO_3 undergoes disproportionation reaction to produce H_3PO_4 and PH_3.

(B) While H_3PO_3 can act as reducing agent, H_3PO_4 cannot.

(C) H_3PO_3 is a monobasic acid.

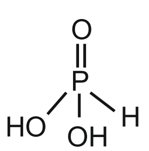
(D) The H atom of P-H bond in H_3PO_3 is not ionizable in water.

Answer (A,B,D)

Sol. $4H_3PO_3 \xrightarrow{\Delta} PH_3 + 3H_3PO_4$

In H_3PO_4, phosphorous is present in highest oxidation state, i.e., +5. So H_3PO_4 cannot acts as reducing agent.

Structure of H_3PO_3,



It is a dibasic acid.

H atom present in P–H bond is not ionizable.

These P-H bonds are not ionisable to give H^+ and do not play any role in basicity. Only those H atoms which are attached with oxygen in P-OH form are ionisable and cause the basicity. Thus, H_3PO_3 and H_3PO_4 are dibasic and tribasic, respectively as the structure of H_3PO_3 has two P–OH bonds and H_3PO_4 three.

SECTION - 2

- This section contains **THREE (03)** question stems.
- There are TWO (02) questions corresponding to each question stem.
- The answer to each question is a NUMERICAL VALUE.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +2 If ONLY the correct numerical value is entered at the designated place.

Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 7 and 8

Question Stem

At 298 K, the limiting molar conductivity of a weak monobasic acid is 4×10^2 S cm^2 mol^{-1}. At 298 K, for an aqueous solution of the acid the degree of dissociation is α and the molar conductivity is $y \times 10^2$ S cm^2 mol^{-1}. At 298 K, upon 20 times dilution with water, the molar conductivity of the solution becomes $3y \times 10^2$ S cm^2 mol^{-1}.

7. The value of α is _______.

Answer (0.215)

8. The value of y is _______.

Answer (0.86)

Sol. Solution of Question Nos. 7 and 8

Molar conductivity of HX at infinite dilution

$\Lambda_m^\infty = 4 \times 10^2$ S cm^2 mol^{-1}

Molar conductivity of HX at conc. $c_1 = y \times 10^2$ S cm^2 mol^{-1}

$$\alpha_1 = \frac{\Lambda_m^{c_1}}{\Lambda_m^\infty} = \frac{y \times 10^2}{4 \times 10^2} = \frac{y}{4}$$

On 20 times dilution of the solution of HX

$$\alpha_2 = \frac{\Lambda_m^{c_2}}{\Lambda_m^\infty} = \frac{3y \times 10^2}{4 \times 10^2} = \frac{3y}{4} \qquad \left[c_2 = \frac{c_1}{20}\right]$$

$$\frac{\alpha_1}{\alpha_2} = \frac{1}{3} \qquad \Rightarrow \alpha_2 = 3\alpha_1$$

$$\begin{array}{ccccc} HX & \rightleftharpoons & H^+ & + & X^- \\ c_1(1-\alpha_1) & & c_1\alpha_1 & & c_1\alpha_1 \end{array}$$

$$K_a = \frac{c_1\alpha_1^2}{1-\alpha_1} = \frac{c_2\alpha_2^2}{1-\alpha_2} = \frac{c_1(3\alpha_1)^2}{20(1-3\alpha_1)}$$

$$\frac{1}{1-\alpha_1} = \frac{9}{20(1-3\alpha_1)}$$

$20 - 60\alpha_1 = 9 - 9\alpha_1 \quad \Rightarrow \quad \alpha_1 = \frac{11}{51} = 0.215$

$y = 4\alpha_1 = 0.86$

Question Stem for Question Nos. 9 and 10

Question Stem

Reaction of x g of Sn with HCl quantitatively produced a salt. Entire amount of the salt reacted with y g of nitrobenzene in the presence of required amount of HCl to produce 1.29 g of an organic salt (quantitatively).

(Use Molar masses (in g mol^{-1}) of H, C, N, O, Cl and Sn as 1, 12, 14, 16, 35 and 119, respectively).

9. The value of x is ________.

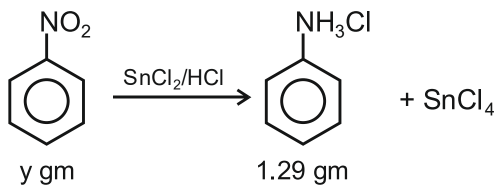
Answer (3.57)

10. The value of y is ________.

Answer (1.23)

Sol. Solution of Question Nos. 9 and 10

$Sn + HCl \rightarrow SnCl_2$



$\Rightarrow$ Moles of ammonium salt = $\frac{1.29}{129}$ = 0.01

$\Rightarrow$ Moles of nitrobenzene = 0.01

$\Rightarrow$ y = 0.01 × Molar mass of nitrobenzene

= 0.01 × 123

y = 1.23

Also

No. of eq. of nitrobenzene = No. of eq. of $SnCl_2$

$6 \times (0.01) = 2 \times n_{SnCl_2}$

$n_{SnCl_2} = 0.03$

$\Rightarrow n_{Sn} = 0.03$

$w_{Sn} = 0.03 \times 119$

x = 3.57

Question Stem for Question Nos. 11 and 12

Question Stem

A sample (5.6 g) containing iron is completely dissolved in cold dilute HCl to prepare a 250 mL of solution. Titration of 25.0 mL of this solution requires 12.5 mL of 0.03 M $KMnO_4$ solution to reach the end point. Number of moles of Fe^{2+} present in 250 mL solution is x × 10^{-2} (consider complete dissolution of $FeCl_2$). The amount of iron present in the sample is y% by weight.

(Assume: $KMnO_4$ reacts only with Fe^{2+} in the solution

Use: Molar mass of iron as 56 g mol^{-1})

11. The value of x is ________.

Answer (1.875)

12. The value of y is ________.

Answer (18.75)

Sol. Solution of Question Nos. 11 and 12

$8H^+ + 5Fe^{2+} + MnO_4^- \rightarrow 5Fe^{3+} + Mn^{2+} + 4H_2O$

For 25 ml,

meq of Fe^{2+} = meq of MnO_4^-

$= 12.5 \times 0.03 \times 5$

For 250 ml,

$$\text{mmoles of } Fe^{2+} = \frac{12.5 \times 0.03 \times 5 \times 250}{25}$$

$$\text{moles of } Fe^{2+} = \frac{18.75}{1000} \text{ mol}$$

$= 18.75 \times 10^{-3}$ mol

$= 1.875 \times 10^{-2}$ mol

x = 1.875

Weight of Fe^{2+} = $1.875 \times 10^{-2} \times 56 = 1.05$ g

$$\% \text{ purity of } Fe^{2+} = \frac{1.05}{5.6} \times 100 = 18.75\%$$

y = 18.75%

SECTION - 3

- This section contains **TWO (02) paragraphs**. Based on each paragraph, there are **TWO (02)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+3	If ONLY the correct option is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	–1	In all other cases.

Paragraph

The amount of energy required to break a bond is same as the amount of energy released when the same bond is formed. In gaseous state, the energy required for homolytic cleavage of a bond is called Bond Dissociation Energy (BDE) or Bond Strength. BDE is affected by *s*-character of the bond and the stability of the radicals formed. Shorter bonds are typically stronger bonds. BDEs for some bonds are given below:

$$H_3C - H(g) \longrightarrow H_3\dot{C}(g) + \dot{H}(g) \quad \Delta H^\circ = 105 \text{ kcal mol}^{-1}$$

$$Cl - Cl(g) \longrightarrow \dot{Cl}(g) + \dot{Cl}(g) \quad \Delta H^\circ = 58 \text{ kcal mol}^{-1}$$

$$H_3C - Cl(g) \longrightarrow H_3\dot{C}(g) + \dot{Cl}(g) \quad \Delta H^\circ = 85 \text{ kcal mol}^{-1}$$

$$H - Cl(g) \longrightarrow \dot{H}(g) + \dot{Cl}(g) \quad \Delta H^\circ = 103 \text{ kcal mol}^{-1}$$

13. Correct match of the **C-H** bonds (shown in bold) in Column **J** with their BDE in Column **K** is

Column J	**Column K**
Molecule	**BDE (kcal mol^{-1})**
(P) **H-C**H$(CH_3)_2$	(i) 132
(Q) **H-C**H_2Ph	(ii) 110
(R) **H-C**H=CH_2	(iii) 95
(S) **H-C**≡CH	(iv) 88

(A) P – iii, Q – iv, R – ii, S – i

(B) P – i, Q – ii, R – iii, S – iv

(C) P – iii, Q – ii, R – i, S – iv

(D) P – ii, Q – i, R – iv, S – iii

Answer (A)

Sol. $H - CH(CH_3)_2 \longrightarrow H\dot{C}(CH_3) - CH_3$

$H - CH_2Ph \longrightarrow {}^\bullet CH_2 - Ph$
Benzyl radical

$H - CH = CH_2 \rightarrow {}^\bullet CH = CH_2$

$H - C \equiv CH \rightarrow {}^\bullet C \equiv CH$

Order of stability of free radical

Q > P > R > S

Stability of free radical $\propto \dfrac{1}{\text{Bond energy}}$

$\therefore$ Order of bond energy :

S > R > P > Q

14. For the following reaction

$$CH_4(g) + Cl_2(g) \xrightarrow{\text{light}} CH_3Cl(g) + HCl(g)$$

the correct statement is

(A) Initiation step is exothermic with $\Delta H° = –58\ kcal\ mol^{–1}$

(B) Propagation step involving $^{\bullet}CH_3$ formation is exothermic with $\Delta H° = –2\ kcal\ mol^{–1}$

(C) Propagation step involving CH_3Cl formation is endothermic with $\Delta H° = +27\ kcal\ mol^{–1}$

(D) The reaction is exothermic with $\Delta H° = –25\ kcal\ mol^{–1}$

Answer (D)

Sol. (1) $Cl_2 \rightarrow 2Cl^{\bullet}$ (Initiation step) $\Delta H = 58$ kcal/mol

(2) $CH_4 + Cl^{\bullet} \rightarrow {}^{\bullet}CH_3 + HCl$

(3) ${}^{\bullet}CH_3 + Cl_2 \rightarrow CH_3Cl + Cl^{\bullet}$

(2)–(3): Propagation step

Step (1) $\rightarrow$ Endothermic (bond breaking)

Step (2) $\rightarrow \Delta H = 105 – 103$

$= 2$ kcal/mol (Endothermic)

Step (3) $\rightarrow \Delta H = 58 – 85$

$= –27$ kcal/mol (Exothermic)

For complete reaction

$$CH_4(g) + Cl_2(g) \xrightarrow{\text{light}} CH_3Cl(g) + HCl(g)$$

$\Delta H = 58 + 105 – 85 – 103$

$= –25$ kcal/mol

Paragraph

The reaction of $K_3[Fe(CN)_6]$ with freshly prepared $FeSO_4$ solution produces a dark blue precipitate called Turnbull's blue. Reaction of $K_4[Fe(CN)_6]$ with the $FeSO_4$ solution in complete absence of air produces a white precipitate X, which turns blue in air. Mixing the $FeSO_4$ solution with $NaNO_3$, followed by a slow addition of concentrated H_2SO_4 through the side of the test tube produces a brown ring.

15. Precipitate X is

(A) $Fe_4[Fe(CN)_6]_3$

(B) $Fe[Fe(CN)_6]$

(C) $K_2Fe[Fe(CN)_6]$

(D) $KFe[Fe(CN)_6]$

Answer (C)

16. Among the following, the brown ring is due to the formation of

(A) $[Fe(NO)_2(SO_4)_2]^{2–}$

(B) $[Fe(NO)_2(H_2O)_4]^{3+}$

(C) $[Fe(NO)_4(SO_4)_2]$

(D) $[Fe(NO)(H_2O)_5]^{2+}$

Answer (D)

Sol. Solution of Question Nos. 15 and 16

$$Fe^{2+} + K_3[Fe(CN)_6] \rightarrow \underset{\text{Turnbull's blue ppt.}}{Fe_3[Fe(CN)_6]_2} \downarrow$$

$$Fe^{2+} + K_4[Fe(CN)_6] \xrightarrow[\text{of air}]{\text{in absence}} \underset{\text{White ppt. (X)}}{K_2Fe[Fe(CN)_6]} \downarrow$$

In air Fe^{2+} gets oxidised to Fe^{3+}

$$Fe^{3+} + [Fe(CN)_6]^{4-} \longrightarrow \underset{\text{Prussian blue}}{Fe_4[Fe(CN)_6]_3} \downarrow$$

$$2NO_3^- + 4H_2SO_4 + 6Fe^{2+} \rightarrow 6Fe^{3+} + 2NO\uparrow + 4SO_4^{2-} + 4H_2O$$

$$[Fe(H_2O)_6]^{2+} + NO \longrightarrow \underset{\text{Compound responsible for brown ring}}{[Fe(H_2O)_5NO]^{2+}} + H_2O$$

$\therefore$ X = $K_2Fe[Fe(CN)_6]$

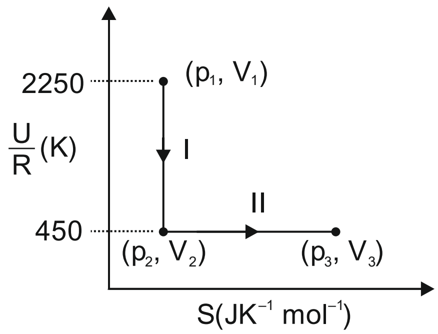
Brown ring is due to $[Fe(H_2O)_5NO]^{2+}$

SECTION - 4

- This section contains THREE (03) questions.
- The answer to each question is a NON-NEGATIVE INTEGER.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If ONLY the correct integer is entered;
Zero Marks	:	0	In all other cases.

17. One mole of an ideal gas at 900 K, undergoes two reversible processes, I followed by II, as shown below. If the work done by the gas in the two processes are same, the value of $\ln\frac{V_3}{V_2}$ is ____.



(U: internal energy, S: entropy, p: pressure, V: volume, R: gas constant)

(Given: molar heat capacity at constant volume, $C_{V, m}$ of the gas is $\frac{5}{2}R$)

Answer (10)

Sol. Process I is adiabatic reversible

Process II is reversible isothermal process

Process I - (Adiabatic Reversible)

$$\frac{\Delta U}{R} = 450 - 2250$$

$\Delta U = -1800R$

$W_I = \Delta U = -1800R$

Process II - (Reversible Isothermal Process)

$T_1 = 900$ K

Calculation of T_2 after reversible adiabatic process

$-1800R = nC_v(T_2 - T_1)$

$$-1800R = 1 \times \frac{5}{2} R(T_2 - 900)$$

$T_2 = 180$ K

$$W_{II} = -nRT_2 \ln\frac{v_3}{v_2} = W_I$$

$$\Rightarrow \quad -1 \times R \times 180 \ln\frac{v_3}{v_2} = -1800R$$

$$\ln\frac{v_3}{v_2} = 10$$

18. Consider a helium (He) atom that absorbs a photon of wavelength 330 nm. The change in the velocity (in cm s^{-1}) of He atom after the photon absorption is_____.

(Assume: Momentum is conserved when photon is absorbed.

Use: Planck constant = 6.6×10^{-34} J s, Avogadro number = 6×10^{23} mol^{-1}, Molar mass of He = 4 g mol^{-1})

Answer (30)

Sol. Momentum of photon $= \frac{h}{\lambda} = \frac{6.6 \times 10^{-27}}{330 \times 10^{-7}}$ gm cm s^{-1}

Momentum of 1 mole of He-atoms = $m\Delta v$

$$\therefore \quad m\Delta v = N_A \times \frac{h}{\lambda}$$

$$4 \times \Delta v = \frac{6 \times 10^{23} \times 6.6 \times 10^{-27}}{330 \times 10^{-7}}$$

$$\Delta v = \frac{6 \times 6.6 \times 10^2}{33 \times 4} = 30 \text{ cm s}^{-1}$$

$\therefore$ Change in velocity of He-atoms = 30 cm s^{-1}

19. Ozonolysis of ClO_2 produces an oxide of chlorine. The average oxidation state of chlorine in this oxide is ____.

Answer (6)

Sol. ClO_2 contains an odd electron and is paramagnetic. It reacts with ozone to give O_2 and Cl_2O_6.

$2ClO_2 + 2O_3 \rightarrow Cl_2O_6 + 2O_2$

In Cl_2O_6, the average oxidation state of Cl is +6.

PART-III : MATHEMATICS

SECTION - 1 (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +4	If only (all) the correct option(s) is(are) chosen;
Partial Marks	: +3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	: 0	If unanswered;
Negative Marks	: – 2	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2 marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 mark;
  choosing ONLY (B) will get +1 mark;
  choosing ONLY (D) will get +1 mark;
  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
  choosing any other option(s) will get “2 marks.

1. Let $S_1 = \{(i, j, k) : i, j, k \in \{1, 2, ..., 10\}\}$,

$S_2 = \{(i, j) : 1 \le i < j + 2 \le 10, i, j \in \{1, 2, ..., 10\}\}$,

$S_3 = \{(i, j, k, l) : 1 \le i < j < k < l, i, j, k, l \in \{1, 2, ..., 10\}\}$,

and $S_4 = \{(i, j, k, l) :$ i, j, k and l are distinct elements in $\{1, 2, ..., 10\}\}$.

If the total number of elements in the set S_r is n_r, r = 1, 2,3,4, then which of the following statements is (are) TRUE?

(A) $n_1 = 1000$ (B) $n_2 = 44$

(C) $n_3 = 220$ (D) $\frac{n_4}{12} = 420$

Answer (A,B,D)

Sol. Number of elements in S_1 = 10 × 10 × 10 = 1000

Number of elements in S_2 = 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 = 44

Number of elements in S_3 = $^{10}C_4$ = 210

Number of elements in S_4 = $^{10}P_4$ = 210 × 4! = 5040

2. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) TRUE ?

(A) $\cos P \geq 1 - \frac{p^2}{2qr}$

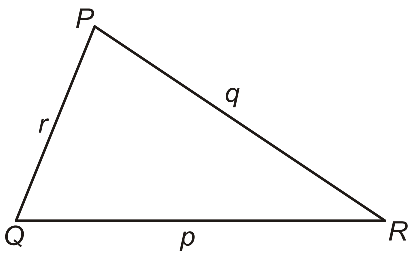
(B) $\cos R \geq \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$

(C) $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) If $p < q$ and $p < r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

Answer (A,B)

Sol.



$$\cos P = \frac{q^2 + r^2 - P^2}{2qr} \quad \text{and} \quad \frac{q^2 + r^2}{2} \geq \sqrt{q^2 . r^2} \quad (\text{AM} \geq \text{GM})$$

$\Rightarrow \; q^2 + r^2 \geq 2qr$

So, $\cos P \geq \frac{2qr - p^2}{2qr}$

$\cos P \geq 1 - \frac{p^2}{2qr}$ (A)

(B) $\frac{(q-r)\cos P + (p-r)\cos Q}{p+q} = \frac{(q\cos P + p\cos Q) - r(\cos P + \cos Q)}{p+q}$

$$= \frac{r(1 - \cos P - \cos Q)}{p+q} = \frac{r(q - p\cos R) - (p - q\cos R)}{p+q} = \frac{(r-p-q) + (p+q)\cos R}{p+q}$$

$$= \cos R + \frac{r-q-p}{p+q} \leq \cos R \quad (\because r < p+q)$$

(C) $\frac{q+r}{p} = \frac{\sin Q + \sin R}{\sin P} \geq \frac{2\sqrt{\sin Q \cdot \sin R}}{\sin P}$

(D) If $p < q$ and $q < r$

So, p is the smallest side, therefore one of Q or R can be obtuse

So, one of $\cos Q$ or $\cos R$ can be negative

Therefore $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$ cannot hold always.

3. Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ be a continuous function such that $f(0)=1$ and $\int_0^{\frac{\pi}{3}} f(t)dt=0$

Then which of the following statements is (are) TRUE?

(A) The equation $f(x) - 3\cos 3x = 0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(B) The equation $f(x) - 3\sin 3x = -\frac{6}{\pi}$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$

(D) $\lim\limits_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

Answer (A,B,C)

Sol. $f(0)=1$, $\int_0^{\frac{\pi}{3}} f(t)dt=0$

(A) Consider a function $g(x)=\int_0^x f(t)dt-\sin 3x$

$g(x)$ is continuous and differentiable function

and $g(0) = 0$

$g\left(\frac{\pi}{3}\right)=0$

By Rolle's theorem $g'(x) = 0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

$f(x)-3\cos 3x=0$ for some $x\in\left(0,\frac{\pi}{3}\right)$

(B) Consider a function

$h(x)=\int_0^x f(t)dt+\cos 3x+\frac{6}{\pi}x$

$h(x)$ is continuous and differentiable function

and $h(0) = 1$

$h\left(\frac{\pi}{3}\right)=1$

By Rolle's theorem $h'(x) = 0$ for at least one $x\in\left(0,\frac{\pi}{3}\right)$

$f(x)-3\sin 3x+\frac{6}{\pi}=0$ for some $x\in\left(0,\frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}$, $\left(\frac{0}{0}\text{form}\right)$

By L' Hopital rule

$$\lim_{x\to 0}\frac{xf(x)+\int_0^x f(t)dt}{-2xe^{x^2}},\left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to 0}\frac{xf'(x)+f(x)+f(x)}{-4x^2e^{x^2}-2e^{x^2}}=\frac{0+2f(0)}{-0-2}=-1$$

(D) $$\lim_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2},\left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to 0}\frac{\sin x\cdot f(x)+\cos x\int_0^x f(t)dt}{2x}$$

$$=\lim_{x\to 0}\frac{\left(\cos x.f(x)+\sin x.f'(x)+\cos x.f(x)-\sin x.\int_0^x f(t)dt\right)}{2}$$

$$=\frac{1+0+1-0}{2}$$

$$=1$$

4. For any real numbers α and β, let $y_{\alpha,\beta}(x)$, $x\in\mathbb{R}$, be the solution of the differential equation $\frac{dy}{dx}+\alpha y=xe^{\beta x}, y(1)=1$

Let $S=\{y_{\alpha,\beta}(x):\alpha,\beta\in\mathbb{R}\}$. Then which of the following functions belong(s) to the set S?

(A) $f(x)=\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}$

(B) $f(x)=-\frac{x^2}{2}e^{-x}+\left(e+\frac{1}{2}\right)e^{-x}$

(C) $f(x)=\frac{e^x}{2}\left(x-\frac{1}{2}\right)+\left(e-\frac{e^2}{4}\right)e^{-x}$

(D) $f(x)=\frac{e^x}{2}\left(\frac{1}{2}-x\right)+\left(e+\frac{e^2}{4}\right)e^{-x}$

Answer (A, C)

Sol. $\frac{dy}{dx}+\alpha y=xe^{\beta x}$

Integrating factor (I.F.) = $e^{\int\alpha dx}=e^{\alpha x}$

So, the solution is $y\cdot e^{\alpha x}=\int xe^{\beta x}\cdot e^{\alpha x}dx$

$$ye^{\alpha x}=\int xe^{(\alpha+\beta)x}dx$$

If $\alpha+\beta\neq 0$

$$ye^{\alpha x} = x\frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2} + C$$

$$y = \frac{xe^{\beta x}}{(\alpha+\beta)} - \frac{e^{\beta x}}{(\alpha+\beta)^2} + Ce^{-\alpha x}$$

$$y = \frac{e^{\beta x}}{(\alpha+\beta)}\left(x - \frac{1}{\alpha+\beta}\right) + Ce^{-\alpha x} \quad \text{... (i)}$$

Put $\alpha = \beta = 1$ in (i)

$$y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + Ce^{-x}$$

$y(1) = 1$

$$1 = \frac{e}{2} \times \frac{1}{2} + \frac{C}{e} \Rightarrow C = e - \frac{e^2}{4}$$

So, $y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$

If $\alpha + \beta = 0$ & $\alpha = 1$

$$\frac{dy}{dx} + y = xe^{-x}$$

I.F. $= e^x$

$$ye^x = \int x\,dx$$

$$ye^x = \frac{x^2}{2} + C$$

$$y = \frac{x^2}{2}e^{-x} + Ce^{-x}$$

$y(1) = 1$

$$1 = \frac{1}{2e} + \frac{C}{e} \Rightarrow C = e - \frac{1}{2}$$

$$y = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$$

5. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ for some $\lambda > 0$. If $\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \frac{9}{2}$, then which of the following statements is (are) TRUE ?

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

Answer (A,B,C)

Sol. $OA = 2\hat{i} + 2\hat{j} + \hat{k}$

$\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$

$\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$

$\overrightarrow{OB} \times \overrightarrow{OC} = \overrightarrow{OB} \times \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA}) = -\frac{\lambda}{2}\overrightarrow{OB} \times \overrightarrow{OA} = \frac{\lambda}{2}(\overrightarrow{OA} \times \overrightarrow{OB})$

Now, $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 6\hat{i} - 3\hat{j} - 6\hat{k}$

So, $\overrightarrow{OB} \times \overrightarrow{OC} = \frac{3\lambda}{2}(2\hat{i} - \hat{j} - 2\hat{k})$

$\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \left|\frac{9\lambda}{2}\right| = \frac{9}{2}$

So, $\lambda = 1 \quad (\because \lambda > 0)$

$\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$

$\overrightarrow{OC} = \frac{1}{2}(-\hat{i} - 4\hat{j} + \hat{k})$

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ = $\frac{\overrightarrow{OC} \cdot \overrightarrow{OA}}{\left|\overrightarrow{OA}\right|} = \frac{\frac{1}{2}(-2-8+1)}{3} = -\frac{3}{2}$

(B) Area of the triangle OAB = $\frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{9}{2}$

(C) Area of the triangle ABC is = $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -4 & 1 \\ -\frac{5}{2} & -4 & -\frac{1}{2} \end{vmatrix}\right\| = \frac{1}{2}\left|6\hat{i} - 3\hat{j} - 6\hat{k}\right| = \frac{9}{2}$

(D) Acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ $= \theta$

$\frac{(\overrightarrow{OA} + \overrightarrow{OC}) \cdot (\overrightarrow{OA} - \overrightarrow{OC})}{\left|\overrightarrow{OA} + \overrightarrow{OC}\right| \left|\overrightarrow{OA} - \overrightarrow{OC}\right|} = \cos\theta$

$\cos\theta = \frac{\left(\frac{3}{2}\hat{i} + \frac{3}{2}\hat{k}\right) \cdot \left(\frac{5}{2}\hat{i} + 4\hat{j} + \frac{1}{2}\hat{k}\right)}{\frac{3}{2}\sqrt{2} \times \sqrt{\frac{90}{4}}} = \frac{18}{3\sqrt{2}\sqrt{90}}$

$\theta \neq \frac{\pi}{3}$

6. Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) TRUE ?

(A) The triangle PFQ is a right-angled triangle

(B) The triangle QPQ' is a right-angled triangle

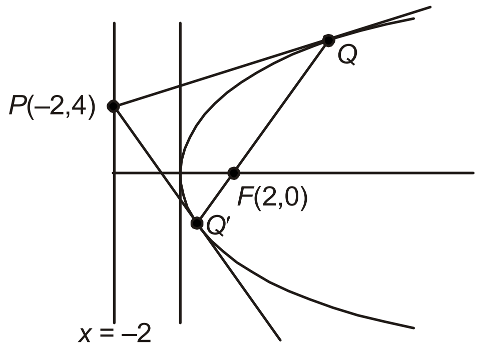
(C) The distance between P and F is $5\sqrt{2}$

(D) F lies on the line joining Q and Q'

Answer (A,B,D)

Sol. $E : y^2 = 8x$

$P : (-2, 4)$



Point P (–2, 4) lies on directrix ($x = -2$) of parabola $y^2 = 8x$

So, $\angle QPQ' = \frac{\pi}{2}$ and chord QQ' is a focal chord and segment PQ subtends right angle at the focus.

Slope of $QQ' = \frac{2}{t_1 + t_2} = 1$

Slope of $PF = -1$

$PF = 4\sqrt{2}$

SECTION - 2 (Maximum Marks : 12)

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+2	If ONLY the correct numerical value is entered.
Zero Marks	:	0	In all other cases.

Question Stem for Question Nos. 7 and 8

Consider the region $R = \{(x, y) \in R \times R : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let F be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in F. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

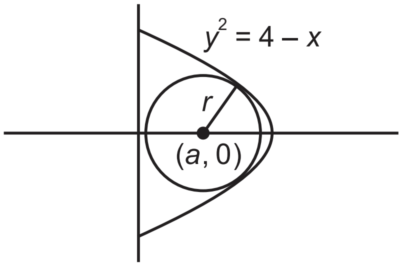
7. The radius of the circle C is _____.

Answer (1.50)

8. The value of α is _____.

Answer (2.00)

Sol. For comprehension Q7 & Q8



Let the circle be,

$(x - a)^2 + y^2 = r^2$

Solving it with parabola

$y^2 = 4 - x$ we get

$(x - a)^2 + 4 - x = r^2$

$\Rightarrow x^2 - x(2a + 1) + (a^2 + 4 - r^2) = 0$ ...(1)

$D = 0$

$\Rightarrow 4r^2 + 4a - 15 = 0$

Clearly $a \geq r$

So $4r^2 + 4r - 15 \leq 0$

$\Rightarrow r_{max} = \frac{3}{2} = a$

Radius of circle C is $\frac{3}{2}$

From (1) $x^2 - 4x + 4 = 0$

$\Rightarrow x = 2 = \alpha$

Question Stem for Question Nos. 9 and 10

Let $f_1 : (0, \infty) \to \mathrm{R}$ and $f_2 : (0, \infty) \to R$ be defined by $f_1(x) = \int_0^x \prod_{j=1}^{21} (t - j)^j \, dt, x > 0$

and $f_2(x) = 98(x - 1)^{50} - 600(x - 1)^{49} + 2450$, $x > 0$, where, for any positive integer n and real number $a_1, a_2, \ldots a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i = 1, 2$, in the interval $(0, \infty)$.

Solution for Q9 and 10

$$f_1'(x) = \prod_{j=1}^{21}(x-j)^j$$

$$f_1'(x) = (x-1)(x-2)^2(x-3)^3,\ldots,(x-20)^{20}(x-21)^{21}$$

Checking the sign scheme of $f_1'(x)$ at x = 1, 2, 3, ..., 21, we get

$f_1(x)$ has local minima at x = 1, 5, 9, 13, 17, 21 and local maxima at x = 3, 7, 11, 15, 19

$\Rightarrow$ $m_1 = 6$, $n_1 = 5$

$f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$

$$f_2'(x) = 98 \times 50(x-1)^{49} - 600 \times 49 \times (x-1)^{48}$$

$= 98 \times 50 \times (x-1)^{48}(x-7)$

$f_2(x)$ has local minimum at $x = 7$ and no local maximum.

$\Rightarrow$ $m_2 = 1$, $n_2 = 0$

9. The value of $2m_1 + 3n_1 + m_1n_1$ is _____.

Answer (57.00)

Sol. $2m_1 + 3n_1 + m_1n_1$

$= 2 \times 6 + 3 \times 5 + 6 \times 5$

$= 57$

10. The value of $6m_2 + 4n_2 + 8m_2n_2$ is _____.

Answer (06.00)

Sol. $6m_2 + 4n_2 + 8m_2n_2$

$= 6 \times 1 + 4 \times 0 + 8 \times 1 \times 0 = 6$

Question Stem for Question Nos. 11 and 12

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R, i = 1, 2$, and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R$ be functions such that $g_1(x) = 1$, $g_2(x) = |4x - \pi|$ and $f(x) = \sin^2 x$, for all $x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$. Define $S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x) \cdot g_i(x)\, dx, i = 1, 2$

11. The value of $\frac{16S_1}{\pi}$ is _____.

Answer (02.00)

Sol. $S_1 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^2 x.1dx$

$$= \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} (1-\cos 2x)\,dx$$

$$=\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$S_1=\frac{1}{2}\left(\frac{\pi}{4}-0\right)=\frac{\pi}{8}$$

$$\Rightarrow \frac{16S_1}{\pi}=2$$

12. The value of $\frac{48S_2}{\pi^2}$ is _____.

Answer (01.50)

Sol. $$S_2=\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}\sin^2 x.|4x-\pi|dx$$

$$=\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}4\sin^2 x\left|x-\frac{\pi}{4}\right|dx$$

Let $x-\frac{\pi}{4}=t\Rightarrow\ dx=dt$

$$S_2=\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}4\sin^2\left(\frac{\pi}{4}+t\right)|t|dt$$

$$=\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}2\left(1-\cos 2\left(\frac{\pi}{4}+t\right)\right)|t|dt$$

$$=\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}(2+2\sin 2t)|t|dt$$

$$=2\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}|t|dt+2\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}|t|\sin(2t)dt$$

$$=4\int_0^{\frac{\pi}{8}}tdt+0$$

$$S_2=2t^2\Big]_0^{\frac{\pi}{8}}=\frac{\pi^2}{32}$$

$$\Rightarrow \frac{48S_2}{\pi^2}=\frac{3}{2}$$

SECTION - 3 (Maximum Marks : 12)

- This section contains **TWO (02) paragraphs**. Based on each paragraph, there are **TWO (02) questions.**
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+3	If ONLY the correct option is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	-1	In all other cases.

Paragraph

Let $M = \{(x, y) \in R \times R : x^2 + y^2 \le r^2\}$, where $r > 0$. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}$, $n = 1, 2, 3, \ldots$. Let $S_0 = 0$ and, for $n \ge 1$, let S_n denote the sum of the first n terms of this progression. For $n \ge 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n, and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n.

13. Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

(A) $k + 2l = 22$

(B) $2k + l = 26$

(C) $2k + 3l = 34$

(D) $3k + 2l = 40$

Answer (D)

Sol. $\because$ $a_n = \frac{1}{2^{n-1}}$ and $S_n = 2\left(1 - \frac{1}{2^n}\right)$

For circles C_n to be inside M.

$$S_{n-1} + a_n < \frac{1025}{513}$$

$$\Rightarrow S_n < \frac{1025}{513}$$

$$\Rightarrow 1 - \frac{1}{2^n} < \frac{1025}{1026} = 1 - \frac{1}{1026}$$

$\Rightarrow$ $2^n < 1026$

$\Rightarrow$ $n \le 10$

$\therefore$ Number of circles inside be 10 = K

Clearly alternate circles do not intersect each other *i.e.*, C_1, C_3, C_5, C_7, C_9 do not intersect each other as well as C_2, C_4, C_6, C_8 and C_{10} do not intersect each other hence maximum 5 set of circles do not intersect each other.

$\therefore$ $l = 5$

$\therefore$ $3K + 2l = 40$

$\therefore$ Option (D) is correct

14. Consider M with $r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

(A) 198 (B) 199

(C) 200 (D) 201

Answer (B)

Sol. $\because \quad r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$

Now, $\sqrt{2}\, S_{n-1} + a_n < \left(\frac{2^{199}-1}{2^{198}}\right)\sqrt{2}$

$2\cdot\sqrt{2}\left(1-\frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}} < \left(\frac{2^{199}-1}{2^{198}}\right)\cdot$

$\therefore \quad 2\sqrt{2}-\frac{\sqrt{2}}{2^{n-2}}+\frac{1}{2^{n-1}} < 2\sqrt{2}-\frac{\sqrt{2}}{2^{198}}$

$\frac{1}{2^{n-2}}\left(\frac{1}{2}-\sqrt{2}\right) < -\frac{\sqrt{2}}{2^{198}}$

$\frac{2\sqrt{2}-1}{2\cdot 2^{n-2}} > \frac{\sqrt{2}}{2^{198}}$

$2^{n-2} < \left(2-\frac{1}{\sqrt{2}}\right)2^{197}$

$\therefore \quad n \le 199$

$\therefore$ Number of circles = 199

Option (B) is correct.

Paragraph

Let $\psi_1 : [0, \infty) \to \mathbb{R}$, $\psi_2 : [0, \infty) \to \mathbb{R}$, $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be functions such that $f(0) = g(0) = 0$,

$\psi_1(x) = e^{-x} + x, \; x \ge 0,$

$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, \; x \ge 0,$

$f(x) = \int_{-x}^{x} \left(|t| - t^2\right) e^{-t^2} dt, \; x > 0$

and $g(x) = \int_0^{x^2} \sqrt{t}\, e^{-t}\, dt, \; x > 0$.

15. Which of the following statements is TRUE?

(A) $f\left(\sqrt{\ln 3}\right) + g\left(\sqrt{\ln 3}\right) = \frac{1}{3}$

(B) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\Psi_1(x) = 1 + \alpha x$

(C) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\Psi_2(x) = 2x\,(\Psi_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

Answer (C)

Sol. $\because \quad g(x)=\int_0^{x^2}\sqrt{t}\, e^{-t}\, dt,\ \ x>0$

Let $t = u^2 \Rightarrow dt = 2u\, du$

$\therefore \quad g(x)=\int_0^{x} u\, e^{-u^2}\cdot 2u\, du$

$$=2\int_0^{x} t^2\, e^{-t^2} dt \qquad \text{...(i)}$$

and $f(x)=\int_{-x}^{x}\left(|t|-t^2\right)e^{-t^2}dt,\ x>0$

$\therefore \quad f(x)=2\int_0^{x}\left(t-t^2\right)e^{-t^2}dt \qquad \text{...(ii)}$

From equation (i) + (ii) : $f(x)+g(x)=\int_0^{x} 2te^{-t^2}\, dt$

Let $t^2 = P \quad \Rightarrow \quad 2t\, dt = dP$

$\therefore \quad f(x)+g(x)=\int_0^{x^2} e^{-P}dP=\left[-e^{-P}\right]_0^{x^2}$

$\therefore \quad f(x)+g(x)=1-e^{-x^2} \qquad \text{...(iii)}$

$\therefore \quad f\left(\sqrt{ln\,3}\right)+g\left(\sqrt{ln\,3}\right)=1-e^{-ln3}=1-\frac{1}{3}=\frac{2}{3}$

$\therefore$ Option (A) is incorrect.

From equation (ii) : $f'(x)=2\left(x-x^2\right)e^{-x^2}=2x(1-x)e^{-x^2}$

$\because$ $f(x)$ is increasing in (0, 1)

$\therefore$ Option (D) is incorrect

$\because$ $\Psi_1(x) = e^{-x} + x$

$\Rightarrow$ $\Psi'_1(x) = 1 - e^{-x} < 1$ for $x > 1$

Then for $\alpha \in (1,\ x)$, $\Psi_1(x)=1+\alpha x$ does not true for $\alpha > 1$.

$\therefore$ Option (B) is incorrect

Now $\psi_2(x)=x^2-2x-2e^{-x}+2$

$\Rightarrow \quad \psi_2'(x)=2x-2+2e^{-x}$

$\therefore \quad \psi_2'(x)=2\psi_1(x)-2$

From LMVT

$\frac{\psi_2(x)-\psi_2(0)}{x-0}=\psi_2'(\beta)$ for $\beta\in(\infty,\ x)$

$\Rightarrow \quad \psi_2(x)=2x(\psi_1(\beta)-1)$

$\therefore$ Option (C) is correct.

16. Which of the following statements is TRUE?

(A) $\Psi_1(x) \le 1$, for all $x > 0$

(B) $\Psi_2(x) \le 0$, for all $x > 0$

(C) $f(x) \ge 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

Answer (D)

Sol. $\because \quad \Psi_1(x) = e^{-x} + x$

and for all x > 0, $\Psi_1(x) > 1$

$\therefore$ (A) is not correct

$\Psi_1(x) = x^2 + 2 - 2\,(e^{-x} + x) > 0$ for $x > 0$

$\therefore$ (B) is not correct

Now, $\sqrt{t}\, e^{-t} = \sqrt{t}\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + ...\infty\right)$

and $\sqrt{t}\, e^{-t} \le t^{\frac{1}{2}} - t^{\frac{3}{2}} + \frac{1}{2}t^{\frac{5}{2}}$

$$\therefore \quad \int_0^{x^2} \sqrt{t}\, e^{-t}\, dt \le \int_0^{x^2} \left(t^{\frac{1}{2}} - t^{\frac{3}{2}} + \frac{1}{2}t^{\frac{5}{2}}\right) dt$$

$$= \frac{2}{3}x^3 - \frac{2}{3}x^5 + \frac{1}{7} + \frac{1}{7}x^7$$

$\therefore$ Option (D) is correct

and $f(x) = \int_{-x}^{x} (|t| - t^2) e^{-t^2}\, dt$

$$= 2\int_0^x (t - t^2) e^{-t^2}\, dt$$

$$= \int_0^x 2te^{-t^2}\, dt - 2\int_0^x t^2 e^{-t^2}\, dt$$

$$= 1 - e^{-x^2} - 2\int_0^x t^2 e^{-t^2}\, dt$$

$$\therefore \quad f(x) \le 1 - e^{-x^2} - 2\int_0^x t^2 \left(1 - t^2\right) dt$$

$$= 1 - e^{-x^2} - 2\frac{x^3}{3} + \frac{2}{5}x^5 \text{ for all } x\left(0, \frac{1}{2}\right)$$

$\therefore$ Option (C) is incorrect.

SECTION - 4 (Maximum Marks : 12)

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If ONLY the correct integer is entered;
Zero Marks	:	0	In all other cases.

17. A number is chosen at random from the set {1, 2, 3......, 2000}. Let p be the probability that the number is a multiple of 3 or a multiple of 7. Then the value of $500p$ is _______.

Answer (214)

Sol. E = a number which is multiple of 3 or multiple of 7

$n(E)$ = (3, 6, 9, ..., 1998) + (7, 14, 21, ..., 1995) – (21, 42, 63, ... 1995)

$n(E)$ = 666+ 285 – 95

$n(E)$ = 856

$n(E)$ = 2000

$$P(E) = \frac{856}{2000}$$

$$P(E) \times 500 = \frac{856}{4} = 214$$

18. Let E be the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. For any three distinct points P, Q and Q' on E, let M (P, Q) be the mid-point of the line segment joining P and Q, and $M(P, Q')$ be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between $M(P, Q)$ and $M(P, Q')$, as P, Q and Q' vary on E, is ______.

Answer (4)

Sol. Let $P(\alpha)$, $Q(\theta)$, $Q'(\theta')$

$$M = \frac{1}{2}(4\cos\alpha + 4\cos\theta),\ \frac{1}{2}(3\sin\alpha + 3\sin\theta)$$

$$M' = \frac{1}{2}(4\cos\alpha + 4\cos\theta'),\ \frac{1}{2}(3\sin\alpha + 3\sin\theta')$$

$$MM' = \frac{1}{2}\sqrt{(4\cos\theta - 4\cos\theta')^2 + (3\sin\theta - 3\sin\theta')^2}$$

$$MM' = \frac{1}{2} \text{ distance between } Q \text{ and } Q'$$

MM' is not depending on P

Maximum of QQ' is possible when QQ' = major axis

QQ' = 2(4) = 8

$$MM' = \frac{1}{2}\cdot(QQ')$$

MM' = 4

19. For any real number x, let $[x]$ denote the largest integer less than or equal to x. If $I = \int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$, then the value of $9I$ is ________.

Answer (182.00)

Sol. $I = \int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$

$y = \frac{10x}{x+1} \quad , \quad 0 \le x \le 10$

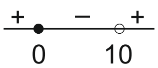
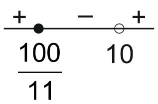
$xy + y = 10x$

$x = \frac{y}{10-y}$

$0 \le \frac{y}{10-y} \le 10$

$\frac{y}{10-y} \ge 0$ and $\frac{y}{10-y} - 10 \le 0$

$\frac{y}{y-10} \le 0$ and $\frac{11y-100}{y-10} \ge 0$

 and 

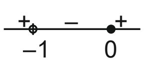
$y \in [0,\ 10)$ and $y \in \left(-\infty, \frac{100}{11}\right] \cup (10, \infty)$

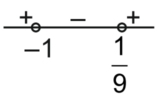
$y \in \left[0, \frac{100}{11}\right]$

$\sqrt{y} \in \left[0, \frac{10}{\sqrt{11}}\right] \quad \Rightarrow \quad \left[\sqrt{y}\right] = \{0, 1, 2, 3\}$

Cose I : $0 \le \frac{10x}{x+1} < 1$

$\frac{10x}{x+1} \ge 0$ and $\frac{10x}{x+1} - 1 < 0$

 and $\frac{9x-1}{x+1} < 0$



$x \in (-\infty, -1) \cup [0,\ \infty)$ and $x \in \left(-1, \frac{1}{9}\right)$

$x \in \left[0, \frac{1}{9}\right)$ then $\left[\sqrt{\frac{10x}{x+1}}\right] = 0$

Case II : $1 \le \dfrac{10x}{x+1} < 4$

$$\frac{10x}{x+1} - 1 \ge 0 \quad \text{and} \quad \frac{10x}{x+1} - 4 < 0$$

$$\frac{9x-1}{x+1} \ge 0 \quad \text{and} \quad \frac{6x-4}{x+1} < 0$$

$$\underset{-1}{+\circ} \; - \; \underset{\frac{1}{9}}{\bullet +} \quad \text{and} \quad \underset{-1}{+\circ} \; - \; \underset{+\frac{2}{3}}{\circ +}$$

$$x \in (-\infty, -1) \cup \left[\frac{1}{9}, \infty\right) \quad \text{and} \quad x \in \left(-1, \frac{2}{3}\right)$$

$$x \in \left[\frac{1}{9}, \frac{2}{3}\right) \quad , \quad \left[\sqrt{\frac{10x}{x+1}}\right] = 1$$

Case III : $4 \le \dfrac{10x}{x+1} < 9$

$$\frac{10x}{x+1} - 4 \ge 0 \quad \text{and} \quad \frac{10x}{x+1} < 9$$

$$\frac{6x-4}{x+1} \ge 0 \quad \text{and} \quad \frac{x-9}{x+1} < 0$$

$$\underset{-1}{+\circ} \; - \; \underset{\frac{2}{3}}{\bullet +} \quad \text{and} \quad \underset{-1}{+\circ} \; - \; \underset{9}{\circ +}$$

$$x \in (-\infty, -1) \cup \left[\frac{2}{3}, \infty\right) \qquad x \in (-1, 9)$$

$$x \in \left[\frac{2}{3}, 9\right) \quad ; \quad \left[\sqrt{\frac{10x}{x+1}}\right] = 2$$

Case IV : $x \in [9, 10] \quad \Rightarrow \quad \left[\sqrt{\dfrac{10x}{x+1}}\right] = 3$

$$I = \int_0^{\frac{1}{9}} 0 \cdot dx + \int_{\frac{1}{9}}^{\frac{2}{3}} 1 \cdot dx + \int_{\frac{2}{3}}^{9} 2 \cdot dx + \int_9^{10} 3 \cdot dx$$

$$I = \left(\frac{2}{3} - \frac{1}{9}\right) + 2\left(9 - \frac{2}{3}\right) + 3(10 - 9)$$

$$I = \frac{5}{9} + \frac{50}{3} + 3$$

$9I = 182$